

# Special rank one groups are perfect

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**Abstract.** We prove that special (abstract) rank one groups with arbitrary unipotent subgroups of size at least 4 are perfect.

**Mathematics Subject Classification (2010).** Primary 20E42; Secondary 51E42.

**Keywords.** special (abstract) rank one group, Moufang set.

## 1. Introduction

J. Tits [3] defined Moufang sets in order to axiomatize the linear algebraic groups of relative rank one. A closely related concept of so-called (abstract) rank one groups has been introduced by F. G. Timmesfeld [2].

Here a group  $X$  is an (abstract) rank one group with unipotent subgroups  $A$  and  $B$ , if  $X = \langle A, B \rangle$  with  $A$  and  $B$  different subgroups of  $X$ , and (writing  $A^b = b^{-1}Ab$ )

for each  $1 \neq a \in A$ , there is an element  $1 \neq b \in B$  such that  $A^b = B^a$ ,  
and vice versa.

We emphasize that in contrast to Timmesfeld's definition [2, p. 1] we do not assume that  $A$  and  $B$  are nilpotent. In an (abstract) rank one group  $X$  with unipotent subgroups  $A$  and  $B$ , the element  $1 \neq b \in B$  with  $A^b = B^a$  is uniquely determined for each  $1 \neq a \in A$  (as  $A \neq B$ ) and denoted by  $b(a)$ . Similarly we define  $a(b)$ .

We say  $X$  is special, if  $b(a^{-1}) = b(a)^{-1}$  for all  $1 \neq a \in A$ . This is equivalent with Timmesfeld's original definition, see Timmesfeld [2, I (2.2), p. 17]. In this note we prove

**Theorem 1.1.** *Any special (abstract) rank one group with arbitrary unipotent subgroups of size at least 4 is perfect.*

By [2, I (1.10), p. 13] this yields that a special (abstract) rank one group with abelian unipotent subgroups is either quasi-simple or isomorphic to  $\mathrm{SL}_2(2)$  or  $(\mathrm{P})\mathrm{SL}_2(3)$ , as was conjectured by Timmesfeld [2, Remark, p. 26].

We remark that  $X/Z(X)$  is the little projective group of a Moufang set and that is the point of view of T. De Medts, Y. Segev and K. Tent. In [1, Theorem 1.12] they prove that the little projective group  $G$  of a special Moufang set  $\mathbb{M}(U, \tau)$  with  $|U| \geq 4$  satisfies  $U_\infty = [U_\infty, G_{0,\infty}]$ . From this they deduce Theorem 1.1 above.

The proof of Theorem 1.1 given below is short, elementary and self-contained. It does not need a case differentiation whether  $A$  is an elementary abelian 2-group or not. We show that  $a_1(A \cap X') = a_2(A \cap X')$  for all  $1 \neq a_1, a_2 \in A$  with  $a_1 a_2 \neq 1$ . (Here  $X' = [X, X]$  is the commutator subgroup of  $X$ .)

## 2. The proof of Theorem 1.1

Let  $X$  be a special (abstract) rank one group with unipotent subgroups  $A$  and  $B$ . For each  $1 \neq a \in A$ , we set  $n(a) := ab(a)^{-1}a$ . Then  $B^{n(a)} = A$ . As  $b(a)^{-1} = b(a^{-1})$ , also  $A^{n(a)} = B$ . Thus  $n(a)n(a') \in H := N_X(A) \cap N_X(B)$ , for all  $1 \neq a, a' \in A$ . For  $1 \neq a \in A$ , we have

$$B^{a^{-1}n(a)} = B^a \quad (2.1)$$

**Lemma 2.1.** *We have  $B^{a_1 a_2 n(a_2^{-1}) a_2} = A^{b(a_1)b(a_2)}$ , for all  $1 \neq a_1, a_2 \in A$ .*

*Proof.* By the definition of  $n(a_2^{-1})$  we have  $B^{a_1 a_2 n(a_2^{-1}) a_2} = B^{a_1 b(a_2^{-1})^{-1}}$ . As  $X$  is special, the left hand side of the claim equals  $A^{b(a_1)b(a_2)}$ .  $\square$

**Lemma 2.2.** *We have  $a_1 \in a_2(A \cap X')$ , for all  $1 \neq a_1, a_2 \in A$  with  $a_1 a_2 \neq 1$ .*

*Proof.* Let  $1 \neq a_1, a_2 \in A$  with  $a_1 a_2 \neq 1$ . We set  $h := n(a_1 a_2)n(a_2^{-1}) \in H$ . By Lemma 2.1 and (2.1) we have  $R := A^{b(a_1)b(a_2)} = B^{a_2^{-1} a_1^{-1} h a_2}$ . As  $a_2^{-1} a_1^{-1} h a_2 = h a_1^{-1} [a_1^{-1}, a_2] [a_2^{-1} a_1^{-1} a_2, h] [h, a_2]$  and  $B^h = B$ , we obtain that  $R = B^{a_1^{-1} a_0}$  with  $a_0 \in A \cap X'$ .

Note that  $b(a_1)b(a_2) = b(a_2)b(a_3)$ , where  $1 \neq a_3 = a(b_3) \in A$  with  $1 \neq b_3 = b(a_2)^{-1}b(a_1)b(a_2) \in B$ . Necessarily  $a_2 a_3 \neq 1$ . Otherwise Lemma 2.1 implies that  $R = A^{b(a_1)b(a_2)} = A^{b(a_2)b(a_3)} = B^{n(a_3^{-1})a_3} = A$ , a contradiction as  $R = B^a$  with  $a \in A$ . As above we have  $R = A^{b(a_2)b(a_3)} = B^{a_2^{-1} a_4}$  with  $a_4 \in A \cap X'$ . As  $N_A(B) = 1$ , we obtain  $a_1^{-1}(A \cap X') = a_2^{-1}(A \cap X')$ . Thus the claim holds.  $\square$

When  $A \cap X' = 1$ , then Lemma 2.2 implies that  $A \subseteq \{1, a, a^{-1}\}$ , where  $1 \neq a \in A$ ; i.e.,  $|A| \leq 3$ . Thus for  $|A| \geq 4$ , we may choose  $1 \neq a \in A \cap X'$ . By Lemma 2.2, we obtain  $A \subseteq a(A \cap X') \cup \{1, a^{-1}\} \subseteq A \cap X'$ , as desired.

## References

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